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Semisimple Rings and Von Neumann Regular Rings of Generalized Power Series

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DEDICATED TO MY FRIEND CHRISTIAN JENSEN, AT THE BEGINNING OF HIS SECOND YOUTH

In this paper we continue our investigation of generalized power series. The main theorem determines rings of generalized power series which are Von Neumann regular rings and semisimple rings. In the final section we give a new proof of Neumann's theorem on skewfields of generalized power series with totally ordered group of exponents. Using a result of Erdös and Radò, we deduce a simple proof of a theorem in [1], which is proven here also for skewfields.

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1. ORDERED SETS

The ordered set $(S, \leq)$ is said to be *artinian* (resp. *noetherian*, resp. *narrow*) if it does not contain any infinite strictly descending subsets (resp. any infinite strictly ascending subsets, resp. any infinite trivially ordered subsets). A set which is artinian and narrow is also called a *quasi-well-ordered* set.

A totally ordered set is artinian if and only if it is well-ordered. The following facts are well-known:

(1.1) *An ordered set* $(S, \leq)$ *is finite if and only if it is noetherian, artinian, and narrow.*

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(1.2) *Let* $(S, \leq)$ *be an ordered set. The following conditions are equivalent*:

(1) $(S, \leq)$ *is artinian and narrow.*

(2) *If* $s_1, s_2, \dots \in S$ *there exist natural numbers* $i_1 < i_2 < \cdots$ *such that* $s_{i_1} < s_{i_2} \leq \cdots$.

(3) *If* $s_1, s_2, \dots \in S$ *there exist natural numbers* $i < j$ *such that* $s_i < s_j$.

(1.3) *Let* $(S_1, \leq_1), \dots, (S_k, \leq_k)$ *be ordered sets,* $S = \prod_{i=1}^{k} S_i$ *endowed with the product order* $\leq$. *Then* $(S, \leq)$ *is artinian and narrow if and only if each* $(S_i, \leq_i)$ *is artinian and narrow.*

We note that if $\leq$, $\leq'$ are orders on S such that $s \leq t$ implies $s \leq' t$ for any $s, t \in S$, and if T is a subset of S which is artinian and narrow with respect to $\leq$, then it is also artinian and narrow with respect to $\leq'$.

2. ORDERED MONOIDS

Let $(S, +)$ be a monoid with a neutral element 0. Let $\leq$ be an order relation on S. $(S, +, \leq)$ is an *ordered monoid* if the following conditions are satisfied: if $s \leq s'$ then $s + t \leq s' + t$ and $t + s \leq t + s'$ for all $s, s', t \in S$. The order is *strict* when $s < s'$ implies $s + t < s' + t$ and $t + s < t + s'$ for all $s, s', t \in S$. A monoid S endowed with a compatible strict total order is *cancellative* if $s + t = s' + t$ or if $t + s = t + s'$ then $s = s'$.

We shall require the following well-known fact:

(2.1) *Let* $T_1, \dots, T_k$ $(k \geq 1)$ *be artinian and narrow subsets of* $(S, +, \leq)$. *Then* $T_1 + \cdots + T_k$ *is artinian and narrow.*

Neumann proved:

(2.2) *Let* $(S, +, \leq)$ *be a totally ordered commutative monoid,* T *a well-ordered subset of* S *such that* $0 < t$ *for every* $t \in T$. *Then the monoid* $\langle T \rangle = \bigcup_{n=1}^{\infty} nT$ *generated by* T *is also a well-ordered subset of* S (here $nT = \{t_1 + \cdots + t_n | t_i \in T$ for each $i\}$).

This result was extended by Erdös and Radò (see Rosenstein [9]):

(2.3) *Let* $(S, +, \leq)$ *be a commutative non-trivially ordered monoid and let* T *be an artinian and narrow subset of* S *such that* $0 < t$ *for every* $t \in T$. *Then* $\langle T \rangle = \bigcup_{n=1}^{\infty} nT$ *is artinian and narrow.*

The order $\leq$ on S is said to be *subtotal* when for any $s, t \in S$ there exists an integer $k \geq 1$ such that $ks \leq kt$ or $kt \leq ks$. If S is commutative group, the order $\leq$ is subtotal if and only if for every $s \in S$ there exists $k \geq 1$ such that $ks \geq 0$ or $ks \leq 0$.

Let $\leq$ be a subtotal order on the commutative monoid S. We define the binary relation $\leq'$ on S by letting $s \leq' t$ whenever there exists an integer $k \geq 1$ such that $ks \leq kt$. If S is a torsion-free commutative monoid, then $\leq'$ is a compatible total order, finer then $\leq$, that is, if $s \leq t$ then $s \leq' t$.

3. REGULAR RINGS

Let R be a ring with unit element 1. R is said to be a (Von Neumann) *regular ring* when for every $x \in R$ there exists $y \in R$ such that $xyx = x$.

A cartesian product of regular rings is a regular ring. Every semisimple ring is regular.

(3.1) *If R is a regular ring and $x \in R$, $x \neq 0$, is not a zero divisor, then x is invertible.*

Proof. Let $x \in R$, $x \neq 0$ and let $y \in R$ be such that $xyx = x$. Then $x(yx - 1) = (xy - 1)x = 0$. So $xy = yx = 1$. ▮

(3.2) *Let R be a ring. R is a skewfield if and only if it is regular and has no zero divisors.*

Proof. One implication is obvious and the other follows at once from (3.1). ▮

We shall require the following theorem:

(3.3) *Let R be a ring. Then R is semisimple if and only if R is regular and every set of non-zero mutually orthogonal idempotents of R is finite.*

Proof. It is well known that if R is semisimple then it is regular and each set of non-zero mutually orthogonal idempotents is finite. The converse is due to Kaplansky (see Goodearl [3, Corollary 2.16]).

4. GENERALIZED POWER SERIES

Let $(S, +, \leq)$ be a strictly ordered monoid and let $(R, +, \cdot)$ be a ring. It is not assumed that S is abelian, nor that R is commutative.

If $f\colon S \to R$ let

$$\operatorname{supp}(f) = \{s \in S \mid f(s) \neq 0\}.$$

Let A be the set of all $f\colon S \to R$ such that $\operatorname{supp}(f)$ is artinian and narrow. A is a subgroup of the additive group of all mappings from S to R with pointwise addition, because $\operatorname{supp}(f+g) \subseteq \operatorname{supp}(f) \cup \operatorname{supp}(g)$ and the union of two artinian and narrow subsets of S is again artinian and narrow. The mapping $0: S \to R$ with $0(s) = 0$ for all $s \in S$ is the neutral element of $(A, +)$.

(4.1) *Let* $s \in S$, $f_1, \dots, f_k \in A$ (with $k \geq 1$). *Then the set*

$$\begin{aligned} X &= X(s, f_1, \dots, f_k) \\ &= \{(t_1, \dots, t_k) \in S^k | s = t_1 + \cdots + t_k,\ f_i(t_i) \neq 0 \text{ for } i = 1, \dots, k\} \end{aligned}$$

is finite.

Proof. It is trivial when $k = 1$, so we assume that $k \geq 2$. If X is infinite, there exists j, $1 \leq j \leq k$, such that the jth projection $\mathrm{pr}_j(X)$ is infinite. By renumbering, we assume that $\mathrm{pr}_1(X)$ is infinite. Since $\mathrm{pr}_1(X) \subseteq \mathrm{supp}(f_1)$, it is artinian and narrow and by (1.1), there exist $t_{11}, t_{12}, \dots \in \mathrm{pr}_1(X)$ with $t_{11} < t_{12} < \cdots$. For each j we choose $(t_{1j}, t_{2j}, \dots, t_{kj}) \in X$. The set $T = \{(t_{2j}, \dots, t_{kj}) | j = 1, 2, \dots\} \subseteq \prod_{i=2}^k \mathrm{supp}(f_i)$, so by (1.3), T is artinian and narrow. By (1.2), there exists $j_1 < j_2 < \cdots$ such that $t_{ij_1} \leq t_{ij_2} \leq \cdots$ (for all $i = 2, \dots, k$). Hence $s = t_{1j_1} + t_{2j_1} + \cdots + t_{kj_1} < t_{1j_2} + t_{2j_2} + \cdots + t_{kj_2} = s$, which is absurd. ∎

If $f, g \in A$, we define the mapping $fg: S \to R$ as

$$(fg)(s) = \sum_{(t,u) \in X(s,f,g)} f(t) g(u).$$

Note that there are only finitely many non-zero summands. If $(fg)(s) \neq 0$, then there exists $(t, u) \in X(s, f, g)$ so $s = t + u \in \mathrm{supp}(f) + \mathrm{supp}(g)$. So $\mathrm{supp}(fg) \subseteq \mathrm{supp}(f) + \mathrm{supp}(g)$, thus by (2.1), $\mathrm{supp}(fg)$ is artinian and narrow, hence $fg \in A$. This defines a binary operation of multiplication on A.

It is routine to verify that the operation is associative, right and left distributive with respect to the addition. Moreover, the unit element of A is $e: S \to R$ given by $e(0) = 1$, $e(s) = 0$ for all $s \in S$, $s \neq 0$.

Thus $(A, +, \cdot)$ is a ring, called the *ring of generalized power series*, with coefficients in R and exponents in S. We use the notation $A = [\![R^{S, \leq}]\!]$. The special case where R is a commutative ring and $(S, +, \leq)$ is a strictly ordered monoid has been introduced and studied in several papers (see [6, 7, 1]).

We shall require the following remark. Assume that $\leq, \leq'$ are compatible orders on the monoid S such that if $s \leq t$ then $s \leq' t$; then $A = [\![R^{S, \leq}]\!]$ is a subring of $A' = [\![R^{S, \leq'}]\!]$.

For each $f \in A$, $f \neq 0$, $\mathrm{supp}(f)$ is non-empty artinian and narrow. Let $\pi(f)$ denote the set of minimal elements of $\mathrm{supp}(f)$. If $(S, \leq)$ is totally ordered, then $\pi(f)$ consists of only one element, which is still denoted by $\pi(f)$.

(4.2) *Let* $(S, \leq)$ *be totally ordered. If* $f, g \in A \setminus \{0\}$, *then*:

(1) $\pi(f+g) \geq \min\{\pi(f), \pi(g)\}$. *If* $\pi(f) < \pi(g)$ *then* $\pi(f+g) = \pi(f)$.

(2) $\pi(fg) \geq \pi(f) + \pi(g)$. *If* R *has no zero-divisors, then* $\pi(fg) = \pi(f) + \pi(g)$.

Proof. Let $\pi(f) = s$, $\pi(g) = t$. (1) If $s \leq t$ and $s' < s$ then $f(s') = g(s') = 0$, so $(f+g)(s') = 0$, hence $\pi(f+g) \geq s$. If $s < t$ then $(f+g)(s) = f(s) \neq 0$, so $\pi(f+g) = s$.

(2) Let $u \in \operatorname{supp}(fg)$, so

$$0 \neq (fg)(u) = \sum_{(u', u'') \in X(u, f, g)} f(u')g(u'').$$

So there exist $u' \in \operatorname{supp}(f)$, $u'' \in \operatorname{supp}(g)$, $u' + u'' = u$. So $u' \geq s$, $u'' \geq t$, and $u \geq s + t$, showing that $\pi(f+g) \geq s+t$. Since the order on S is strict and $(u', u'') \in X(s+t, f, g)$, $u' + u'' = s + t$, $u' \geq s$, $u'' \geq t$, hence $u' = s$, $u'' = t$. Thus $(fg)(s+t) = f(s)g(t) \neq 0$, if R has no zero-divisors. ∎

If $n \geq 1$ and R is a ring, let $M_{n\times n}(R)$ denote the ring of all $n \times n$ matrices with entries in R. We have the following canonical isomorphism:

(4.3) $[\![M_{n\times n}(R)^{S,\leq}]\!] \cong M_{n\times n}([\![R^{S,\leq}]\!])$.

Proof. Let $f \in [\![M_{n\times n}(R)^{S,\leq}]\!]$ with $f(s)$ equal to the matrix with the (i,j)-entry denoted by $f(s)_{(i,j)}$. Let $\varphi(f)$ be the matrix with the (i,j)-entry equal to $f_{i,j}\colon S \to R$, defined by $f_{i,j}(s) = f(s)_{(i,j)}$. We have

$$\operatorname{supp}(f) = \bigcup_{i,j=1}^{n} \operatorname{supp}(f_{i,j}),$$

so each $f_{ij} \in [\![R^{S,\leq}]\!]$, so $\varphi(f) \in M_{n\times n}([\![R^{S,\leq}]\!])$.

It is clear that φ is a bijection and also that $\varphi(f+g) = \varphi(f) + \varphi(g)$. Now we verify that $\varphi(fg) = \varphi(f)\varphi(g)$: For every $i, j = 1, \ldots, n$ and $s \in S$ we have

$$\begin{aligned}
(\varphi(fg))_{i,j}(s) &= ((fg)(s))_{(i,j)} \\
&= \left(\sum_{(t,u)\in X(s,f,g)} f(t)g(u) \right)_{(i,j)} \\
&= \sum_{(t,u)\in X(s,f,g)} \left(\sum_{k=1}^{n} f(t)_{(i,k)} g(u)_{(k,j)} \right) \\
&= \sum_{k=1}^{n} \sum_{Y} f(t)_{(i,k)} g(u)_{(k,j)},
\end{aligned}$$

where $Y = \{(t,u) | t+u = s,\ f(t)_{(i,k)} \neq 0,\ g(u)_{(k,j)} \neq 0\}$. From $f(t)_{(i,k)} = \varphi(f)_{i,k}(t)$ and $g(u)_{(k,j)} = \varphi(g)_{k,j}(u)$, we have $Y = X(s, \varphi(f)_{(i,k)}, \varphi(g)_{(k,j)})$. On the other hand

$$\begin{aligned}(\varphi(f)\varphi(g))_{i,j}(s) &= \left(\sum_{k=1}^{n} \varphi(f)_{i,k}\varphi(g)_{k,j}\right)(s)\\ &= \sum_{k=1}^{n}\left(\sum_{(t,u)\in X(s,\,\varphi(f)_{i,k}\varphi(g)_{k,j})} \varphi(f)_{i,k}(t)\,\varphi(g)_{k,j}(u)\right)\\ &= \sum_{k=1}^{n}\sum_{(t,u)\in Y} f(t)_{(i,k)}g(u)_{(k,j)}.\end{aligned}$$

We conclude that $\varphi(fg) = \varphi(f)\varphi(g)$ and φ is a canonical ring isomorphism. ∎

We indicate conditions for A to be a skewfield. Neumann showed, using (2.2):

(4.4) *Assume that the order on S is total. Then A is a skewfield if and only if R is a skewfield and S is a group.*

We shall give in Section 6 another proof of this theorem, not involving (2.2). The above result was extended by Elliott and Ribenboim [1] for the case when R is commutative. The proof involved (2.3) and holds also when R is not commutative:

(4.5) *Assume that S is a commutative monoid. Then A is a skewfield if and only if R is a skewfield, S is a torsion-free group, and the order $\leq$ on S is subtotal.*

5. REGULAR AND SEMISIMPLE RINGS OF GENERALIZED POWER SERIES

We prove our main theorem:

(5.1) THEOREM. *Let R be a ring with unit element* 1 *and let* $(S, +, \leq)$ *be a strictly ordered monoid,* $A = [\![R^{S,\leq}]\!]$. *Assume the following*:

(1°) *There exists an artinian and narrow subset T of S which is infinite* (*hence the order on S is not trivial*).

(2°) *If the order on S is not total, then S is commutative and torsion-free.*

Then the following conditions are equivalent:

(1) *A is regular*.

(2) (a) *S is a group*.

(b) *If S is commutative, the order on S is subtotal*; *if S is non-commutative, the order on S is total*.

(c) *R is a regular ring*.

(d) *Every set of non-zero mutually orthogonal idempotents of R is finite*.

(3) *R is a semisimple ring and conditions* (a) *and* (b) *of* (2) *hold*.

(4) *A is a semisimple ring*.

Proof. **(1)** $\Rightarrow$ **(2)**. (a) S is a group: let $s \in S$ and let $e_s\colon S \to R$ be defined by $e_s(s) = 1$, $e_s(t) = 0$ for all $t \in S$, $t \neq s$. By hypothesis, there exists $g \in A$ such that $e_s = e_s g e_s$. Then $1 = e_s(s) = (e_s g e_s)(s)$. Hence there exists $t \in S$ such that $s + t + s = s$. Since the order is strict, $(S, +)$ is cancellative, so $s + t = t + s = 0$, showing that S is a group.

(b) By hypothesis, if S is not commutative, the order on S is total. Let S be commutative. We show that the order on S is subtotal. Let $s \in S$, $s \neq 0$. Since A is regular, there exists $g \in A$ such that $(e_0 + e_s)g(e_0 + e_s) = e_0 + e_s$. Computing both sides at $0, s, 2s, 3s, \ldots, -s, -2s, -3s, \ldots,$ we have the following relations:

$$
\begin{aligned}
1 &= g(0) + g(-s) + g(-s) + g(-2s)\\
1 &= g(0) + g(0) + g(0) + g(-s)\\
0 &= g(2s) + g(s) + g(s) + g(0)\\
0 &= g(3s) + g(2s) + g(2s) + g(s)\\
0 &= g(4s) + g(3s) + g(3s) + g(2s)\\
0 &= g(5s) + g(4s) + g(4s) + g(3s)\\
&\cdots\\
0 &= g(-s) + g(-2s) + g(-2s) + g(-3s)\\
0 &= g(-2s) + g(-3s) + g(-3s) + g(-4s)\\
0 &= g(-3s) + g(-4s) + g(-4s) + g(-5s)\\
&\cdots
\end{aligned}
$$

Then

(i) $g(0) + g(-s) \neq 0$ or

(ii) $g(-s) + g(-2s) \neq 0$,

and

(i) $g(0)+g(-s)\neq 0$ or

(iii) $g(0)+g(s)\neq 0$.

In case (ii), $g(-s)+g(-2s)\neq 0$, $g(-2s)+g(-3s)\neq 0$, $g(-3s)+g(-4s)\neq 0,\dots$, etc. So there exist infinitely many integers $0<n_1<n_2<n_3<\cdots$ such that $g(-n_is)\neq 0$ so $-n_1s,-n_2s,\ldots\in\operatorname{supp}(g)$. By (1.2) there exist $n_i<n_j$ such that $-n_is\leq -n_js$, so $(n_j-n_i)s\leq 0$.

In case (iii), $g(0)+g(s)\neq 0$, $g(s)+g(2s)\neq 0$, $g(2s)+g(3s)\neq 0,\dots$, so there exist integers $0<n_1<n_2<n_3<\cdots$ such that $g(n_is)\neq 0$ for all $i=1,2,\dots$. So $n_1s,n_2s,\ldots\in\operatorname{supp}(g)$, hence by (1.2) there exist $n_i<n_j$ such that $n_is\leq n_js$, hence $0\leq(n_j-n_i)s$.

If (i) holds but (ii) and (iii) do not hold, then $g(0)+g(s)=0$, $g(s)+g(2s)=0$, $g(2s)+g(3s)=0,\dots$, and $g(-s)+g(-2s)=0$, $g(-2s)+g(-3s)=0$, $g(-3s)+g(-4s)=0,\dots$. If $g(0)\neq 0$, then $0,s,2s,\ldots\in\operatorname{supp}(g)$. Hence, as before, there exist $n_i<n_j$ such that $(n_j-n_i)s\geq 0$. If $g(-s)\neq 0$ then $-s,-2s,-3s,\ldots\in\operatorname{supp}(g)$ and as before there exist $n_i<n_j$ such that $(n_j-n_i)s\leq 0$. This shows the order on S is subtotal.

(c) R is a regular ring: Let $a\in R$, $a\neq 0$ and let $f\colon S\to R$ be given by $f(0)=a$, $f(s)=0$ for every $s\in S$, $s\neq 0$. By hypothesis, there exists $g\in A$ such that $fgf=f$. Then $a=f(0)=(fgf)(0)=ag(0)a$, so R is a regular ring.

(d) (Part 1) We show that there exists an artinian and narrow subset $T\subseteq S$ such that if M is a set of non-zero mutually orthogonal idempotents of R, then $|M|<|T|$. If this is not true, for every artinian and narrow subset T of S, there exists a set M as indicated, with $|T|\leq|M|$. By hypothesis we may choose T to be infinite. There exists an injective map $\theta\colon T\to M$. Let $f\colon S\to R$ be defined by $f(t)=\theta(t)$ for every $t\in T$ and $f(s)=0$ for every $s\in S\setminus T$. Thus $\operatorname{supp}(f)=T$, hence $f\in A$. Since A is regular, there exists $g\in A$ such that $f=fgf$, so $g\neq 0$. For every $t\in T$

$$f(t)=\sum_{(t',u,t'')\in X(t,f,g,f)} f(t')g(u)f(t'').$$

Then $f(t)=f(t)f(t)f(t)=f(t)[\sum f(t')g(u)f(t'')]f(t)$. Since $f(t)f(t')=f(t'')f(t)=0$ when $t'\neq t$, $t''\neq t$, and since $f(t)\neq 0$, $(t,-t,t)\in X(t,f,g,f)$ and $f(t)=f(t)g(-t)f(t)$. Thus $\{-t\mid t\in\operatorname{supp}(f)=T\}\subseteq\operatorname{supp}(g)$, so $\{-t\mid t\in T\}$ is artinian and narrow. Hence T is noetherian; but T is artinian and narrow, so T is finite by (1.1), and this is a contradiction.

(d) (Part 2) Every set M of non-zero mutually orthogonal idempotents of R is finite: Indeed, if this is false, let M_0 be an infinite set of non-zero mutually orthogonal idempotents of R. By Zorn's Lemma, M_0 is contained in a maximal set M of non-zero mutually orthogonal idempotents and M

is infinite. By (d) (Part 1), there exists an artinian and narrow subset T of S such that $|M| < |T|$. Let $\varphi\colon M \to T$ be an injective map and let $f\colon S \to R$ be defined by $f(\varphi(a)) = a$ for all $a \in M$, $f(t) = 0$ for all $t \in S \setminus \varphi(M)$. Then $\operatorname{supp}(f) \subseteq T$ and so $f \in A$.

We show that f is not a zero-divisor: Let $fg = 0$ with $g \in A$, $g \neq 0$. Let $a_0 \in M$ be arbitrary and $t_0 \in \operatorname{supp}(g)$. Then there exists $r \in R$ such that $g(t_0)rg(t_0) = g(t_0)$, so $g(t_0)r$ is a non-zero idempotent. Also

$$0 = (fg)(\varphi(a_0) + t_0) = \sum_{(s,t) \in X(\varphi(a_0)+t_0, f, g)} f(s)g(t).$$

Then $0 = a_0 0 = a_0(\Sigma f(s)g(t)) = a_0 a_0 g(t_0) = a_0 g(t_0)$. Thus $0 = a_0 g(t_0)r$ for every $a_0 \in M$. Then $g(a_0)r \notin M$ and $M \cup \{g(t_0)r\}$ is still a set of non-zero mutually orthogonal idempotents of R, which is absurd, due to the maximality of M.

Since A is regular and f is not a zero-divisor, then f is invertible. Let $h \in A$ be such that $e_0 = fh$. So

$$1 = e_0(0) = \sum_{(s,t) \in X(0, f, h)} f(s)h(t).$$

The set $X(0, f, h)$ is finite. Since M is infinite, there exists $a \in M \setminus \{f(s) | (s,t) \in X(0, f, h)\}$. Then

$$a = a \cdot 1 = a\left(\sum_{(s,t) \in X(0, f, h)} f(s)h(t)\right) = 0.$$

This is an absurdity which completes the proof of **(1)** $\Rightarrow$ **(2)**.

(2) $\Rightarrow$ **(3)**. By (3.3), R is a semisimple ring.

(3) $\Rightarrow$ **(4)**. We have $R = \prod_{i=1}^{k} R_i$ where each R_i is a simple ring. $R_i = M_{n_i \times n_i}(K_i)$, the ring of all $n_i \times n_i$ matrices with entries in a skewfield K_i. Then

$$A = \left[\!\!\left[\left(\prod_{i=1}^{k} R_i\right)^{S, \leq}\right]\!\!\right] = \prod_{i=1}^{k} [\![R_i^{S, \leq}]\!].$$

But $[\![M_{n_i \times n_i}(K_i)^{S, \leq}]\!] \cong M_{n_i \times n_i}[\![K_i^{S, \leq}]\!]$, by (4.3). Since K_i is a skewfield, if the order on the group S is total, by the theorem of Neumann, $L_i = [\![K_i^{S, \leq}]\!]$ is a skewfield. If the order on S is not total, then S is an abelian group, the order on S is subtotal. By (4.5), $L_i = [\![K_i^{S, \leq}]\!]$ is a skewfield. So $A_i = M_{n_i \times n_i}(L_i)$ is a simple ring and $A = \prod_{i=1}^{k} A_i$ is a semisimple ring.

(4) $\Rightarrow$ **(1)**. This is trivial. ∎

We mention explicitly the corollary concerning Laurent series $A = R((X)) = [\![R^{\mathbb{Z}, \leq}]\!]$ ($\leq$ is the usual order on $\mathbb{Z}$).

(5.2) *The following conditions are equivalent*:

(1) *A is a regular ring.*

(2) *R is regular and every set of non-zero mutually orthogonal idempotents of R is finite.*

(3) *R is a semisimple ring.*

(4) *A is a semisimple ring.*

Thus, if K is a field, $R = K^{\mathbb{N}}$ (the ring of sequences of elements in K), then $A = R((X))$ is not a regular ring.

6. SKEWFIELDS OF GENERALIZED POWER SERIES

In this final section we give a new proof of Neumann's theorem (4.4), without applying (2.2). We obtain as a corollary the theorem (4.5), using (2.3).

(6.1) *If $(S, +, \leq)$ is a totally ordered group and R is a skewfield, then $A = [\![R^{S,\leq}]\!]$ is a skewfield.*

Proof. Let $f \in A$, $f \neq 0$, with $\pi(f) = 0$, $f(0) = 1$. We shall prove that f has inverse in A. This suffices to show that A is a skewfield. Indeed, if $g \in A$, $g \neq 0$. Let $\pi(g) = s$, let $h \in A$ be given by $h(-s) = g(s)^{-1}$, $h(t) = 0$ for all $t \in S$, $t \neq -s$. Then $f = gh = hg$ is such that $\pi(f) = 0$, $f(0) = 1$, so f is invertible, hence so is g.

We show that if f is not invertible, if λ is an ordinal such that $|\lambda| > |S|$, then there exists a family $(k_\alpha)_{\alpha<\lambda}$ where each $k_\alpha \in A$, such that, assuming $f_\alpha = e - fk_\alpha \neq 0$, if $\pi(f_\alpha) = s_\alpha$, then $s_\alpha < s_\beta$ for all $\alpha < \beta < \lambda$. This is, however, impossible because $|\lambda| > |S|$. The family $(k_\alpha)_{\alpha<\lambda}$ is defined by transfinite induction.

Let $k_0 = 0$ so $s_0 = \pi(e - fk_0) = 0$. Let μ be an ordinal, $\mu < \lambda$. We assume that k_α has been defined for all $\alpha < \mu$, so that if $s_\alpha = \pi(e - fk_\alpha)$ then $s_\alpha < s_\beta$ when $\alpha < \beta < \mu$.

First Case. $\mu = \nu + 1$. Let $f_\nu = e - fk_\nu$ and let $g_{\nu+1}\colon S \to R$ be defined by $g_{\nu+1}(s_\nu) = f_\nu(s_\nu)$, $g_{\nu+1}(t) = 0$ for all $t \in S$, $t \neq s_\nu$. Let $k_{\nu+1} = k_\nu + g_{\nu+1}$ and $f_{\nu+1} = e - fk_{\nu+1} = f_\nu - fg_{\nu+1}$. So $f_{\nu+1}(s_\nu) = 0$, hence $s_{\nu+1} = \pi(f_{\nu+1}) > s_\nu$.

Second Case. μ is a limit ordinal. We define $k_\mu\colon S \to R$ as follows: $k_\mu(u) = 0$ if $u \geq s_\alpha$ for all $\alpha < \mu$; $k_\mu(u) = k_{\alpha(u)}(u)$ where $\alpha(u) < \mu$, $\alpha(u)$ is the smallest ordinal such that $\mu < s_{\alpha(u)}$. We note that if $\alpha(u) < \beta < \mu$ then $k_{\alpha(u)}(u) = k_\beta(u)$. Indeed, $\pi(k_{\alpha(u)} - k_\beta) = \pi(f(k_{\alpha(u)} - k_\beta)) = \pi((e - fk_\beta) - (e - kf_{\alpha(u)})) = s_{\alpha(u)}$ since $s_{\alpha(u)} < s_\beta$. Since $u < s_{\alpha(u)}$, $k_{\alpha(u)}(u) = k_\beta(u)$.

Next we show that $\operatorname{supp}(k_\mu)$ is well-ordered, so $k_\mu \in A$. Let $U \subseteq \operatorname{supp}(k_\mu)$, $U \neq \emptyset$. If $u \in U$, there exists $\alpha(u) < \mu$ such that $k_\mu(u) = k_{\alpha(u)}(u)$. We consider the non-empty set of ordinals $\{\alpha(u) | u \in U\}$. Let α_1 be its smallest element. Then $U \cap \operatorname{supp}(k_{\alpha_1}) \neq \emptyset$, so it is a well-ordered set and let u_1 be its smallest element. Now we show that $u_1 \leq u$ for every $u \in U$. If $u \in U$ and $\alpha(u) = \alpha_1$, then $u \in U \cap \operatorname{supp}(k_{\alpha_1})$, so $u_1 \leq u$. If $u \in U$ and $\alpha_1 < \alpha(u)$, then $u_1 < s_{\alpha_1} \leq u$.

Let $f_\mu = e - fk_\mu$, $s_\mu = \pi(f_\mu)$. We show that $s_\alpha \leq s_\mu$ for all $\alpha < \mu$; then $s_\alpha < s_\mu$ for all $\alpha < \mu$. If not, there exists $\alpha < \mu$ such that $s_\mu < s_\alpha$. Hence $\pi(k_\alpha - k_\mu) = \pi(f(k_\alpha - k_\mu)) = \pi(f_\mu - k_\alpha) = s_\mu$. But $k_\mu(s_\mu) = k_{\alpha(s_\mu)}(s_\mu) = k_\beta(s_\mu)$ for all $\beta \geq \alpha(s_\mu)$. Then $\alpha < \alpha(s_\mu)$. So $s_\mu < s_\alpha < s_{\alpha(s_\mu)}$. But by definition $\alpha(s_\mu) \leq \alpha$, hence $s_{\alpha(s_\mu)} \leq s_\alpha$ and this is a contradiction.

This concludes the transfinite induction leading to the family $(k_\alpha)_{\alpha < \lambda}$ with $s_\alpha < s_\beta$ for all $\alpha < \beta < \lambda$, which is impossible, thereby proving the theorem. ▮

Now we have:

(6.2) *Let $(S, +, \leq)$ be a torsion-free abelian group with $\leq$ subtotal and let R be a skewfield. Then A is a skewfield.*

Proof. Let $\leq'$ be the total order on S associated to the subtotal order as indicated in Section 2. Let $A' = [\![R^{S, \leq'}]\!]$. By (6.1), A' is a skewfield. As noted in Section 4, $A = [\![R^{S, \leq}]\!]$ is a subring of A'. We show that A is a skewfield. Let $f \in A$, $f \neq 0$ with $\operatorname{supp}(f) \subseteq \{s \in S | 0 \leq' s\}$ and $f(0) = 1$. So there exists $k \in A'$ such that $fk = kf = e$. Then $\operatorname{supp}(k)$ is well-ordered (with respect to $\leq'$) and contained in $\{s \in S | 0 \leq' s\}$, and $k(0) = 1$. We show that $\operatorname{supp}(k)$ is artinian and narrow (with respect to $\leq$), so $k \in A$. For this purpose we prove that $\operatorname{supp}(k) \subseteq \langle \operatorname{supp}(f) \setminus \{0\} \rangle$. If this is not true, there exists $s \in \operatorname{supp}(k)$, smallest (with respect to $\leq'$) such that $s \notin \langle \operatorname{supp}(f) \setminus \{0\} \rangle$. We have

$$0 = e(s) = fk(s) = f(0)k(s) + \sum_{0 < t \leq s} f(t)k(s-t).$$

So

$$0 \neq k(s) = -\sum_{0 < t \leq s} f(t)k(s-t)$$

and there exists t, $0 < t \leq s$, such that $f(t) \neq 0$, $k(s-t) \neq 0$. So $t \in \operatorname{supp}(s) \setminus \{0\}$, $s - t < s$, and $s - t \in \operatorname{supp}(k)$, thus $s - t \in \langle \operatorname{supp}(f) \setminus \{0\} \rangle$, hence $s \in \langle \operatorname{supp}(f) \setminus \{0\} \rangle$ which is a contradiction. By (2.3), $\langle \operatorname{supp}(f) \setminus \{0\} \rangle$ is artinian and narrow, hence $\operatorname{supp}(k)$ is artinian and narrow, so $k \in A$ and f is invertible in A. It follows at once that every $g \in A$, $g \neq 0$, is invertible in A, so A is a skewfield. ▮

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